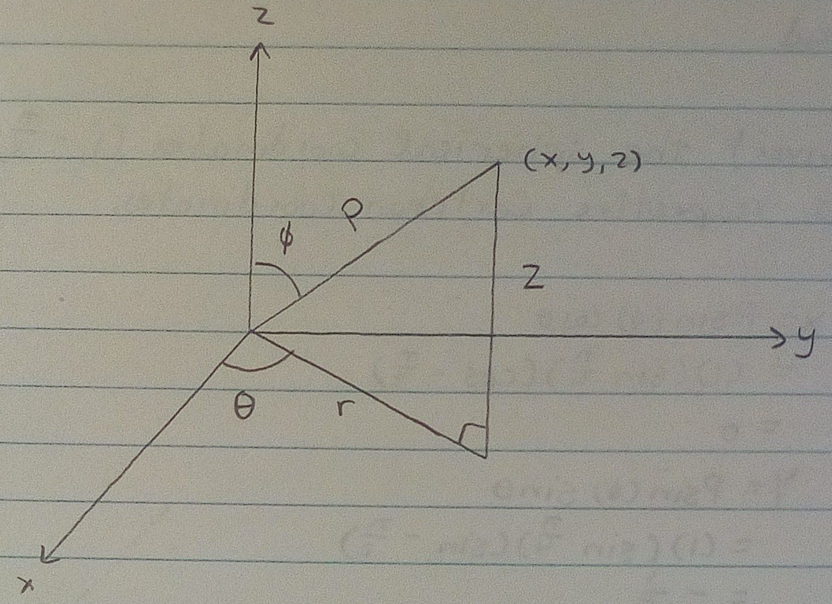
MATB41 Week 3 Notes

1. Spherical Coordinates:



In spherical coordinates, we use (ρ, θ, ϕ) instead of (x, y, z).

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$\left.\begin{aligned} x &= r\cos\theta = \rho\sin\phi\cos\theta \\ y &= r\sin\theta = \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \end{aligned}\right\} \rho \geq 0,\ 0 \leq \theta < 2\pi,\ 0 \leq \phi \leq \pi$$

$$r = \rho\sin\phi$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\phi = \arctan\left(\frac{r}{z}\right) = \arccos\left(\frac{z}{\rho}\right)$$

E.g. 1 Change the cartesian coordinates of $(2, -3, 6)$ into its spherical counterpart.

Solution:

$$\begin{aligned} \rho &= \sqrt{x^2+y^2+z^2} \\ &= \sqrt{(2)^2+(-3)^2+6^2} \\ &= \sqrt{49} \\ &= 7 \end{aligned}$$

$$\begin{aligned} \theta &= 2\pi + \tan^{-1}\left(\frac{y}{x}\right) \\ &= 2\pi + \tan^{-1}\left(\frac{y}{x}\right) \\ &= 2\pi - 0.9828 \\ &\approx 5.300 \text{ rad} \end{aligned}$$

← Since $\tan^{-1}\left(\frac{y}{x}\right)$ is negative, we have to add 2π to it.

$$\phi = \arccos\left(\frac{z}{\rho}\right)$$
$$= \arccos\left(\frac{6}{7}\right)$$
$$\approx 0.541 \text{ rad}$$

E.g. 2 Convert the spherical coordinates $(1, -\frac{\pi}{2}, \frac{\pi}{4})$ to its respective Cartesian coordinates.

Solution: $x = \rho \sin(\phi) \cos\theta$

$$= (1)(\sin \tfrac{\pi}{4})(\cos -\tfrac{\pi}{2})$$
$$= 0$$

$y = \rho \sin(\phi) \sin\theta$

$$= (1)(\sin \tfrac{\pi}{4})(\sin -\tfrac{\pi}{2})$$
$$= -\frac{1}{\sqrt{2}}$$

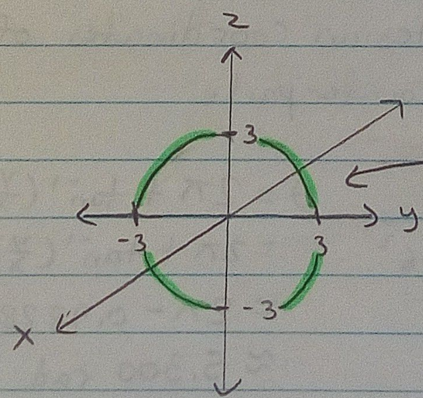
$z = \rho \cos(\phi)$

$$= (1)(\cos \tfrac{\pi}{4})$$
$$= \frac{1}{\sqrt{2}}$$

E.g. 3 Graph $\rho = 3$

Solution: $\rho = \sqrt{x^2+y^2+z^2}$

$$\sqrt{x^2+y^2+z^2} = 3$$
$$x^2+y^2+z^2 = 3^2$$ ← Equation of a sphere with a radius of 3 centered at the origin.



It's suppose to be a sphere and not a circle.

2. Vector Functions:

1. A vector-valued function $f: R^n \to R^m$ is a rule or process that assigns each input x in R^n to its corresponding output y in R^m. m must be greater than 1.

   If $m=1$, then it is a scalar-valued function, also known as a real-valued function.

2. In general, we use the notation $x \mapsto f(x)$ to indicate the value to which a point $x \in R^n$ is sent.

   Similarily, we can write $f: A \subset R^n \to R^m$ to signify that this function, f, takes an element, A, from its domain and maps it to R^m. Here, R^n is the domain of f and R^m is the range of f. If $A \subset R^n$ and $n>1$, then f is a function with several variables.

3. Graphs of Functions:

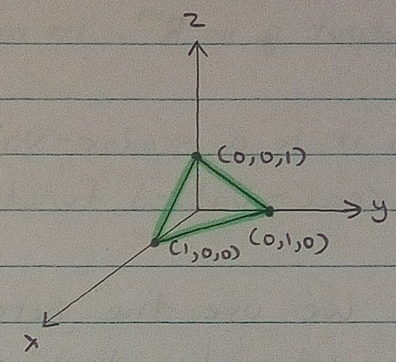
Let $f: U \subset R^n \to R$. The graph of f is defined to be the subset of R^{n+1} consisting of the points $(x_1, x_2, \ldots x_n, f(x_1, \ldots x_n))$ in R^{n+1} for $(x_1, x_2, \ldots x_n)$ in U.

I.e. $f = \{(x_1, x_2, \ldots x_n, f(x_1, \ldots x_n)) \in R^{n+1} \mid (x_1, x_2, \ldots x_n) \in U\}$

E.g. 4 For $f: U \subset R \to R$, its graph is $\{(x, f(x)) \mid x \in U\}$.

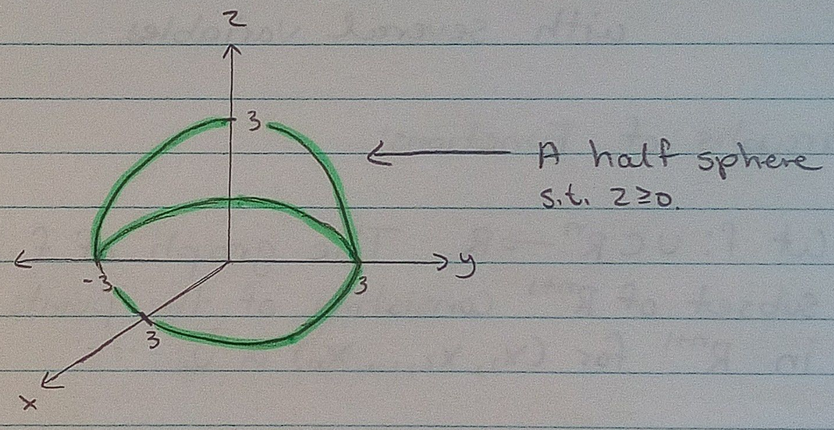
E.g. 5 Graph the function $z = 1-x-y$.

Solution: Rearranging the equation, we get $x+y+z=1$. This is the equation of a plane.



E.g. 6 Graph $z = \sqrt{9-x^2-y^2}$

Solution: Rearranging the equation, we get $x^2+y^2+z^2=3^2$. This is the equation of a sphere. However, because $z=\sqrt{9-x^2-y^2}$ and we know that $\sqrt{\text{anything}}$ is greater than or equal to 0, $z \geq 0$.



4. Level sets, Curves, Surfaces and Contours:

Level set: Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ and let $k \in \mathbb{R}$. The level set of f at value k is defined to be the set of those points $x \in U$ at which $f(x) = k$. ($k \in$ Range (f)).

I.e. Level set: $f = \{(x, k) \mid x \in U\}$.
The level set is always in the domain space.

If $n=2$, we have level curve / level contour.
If $n=3$, we have level surface.

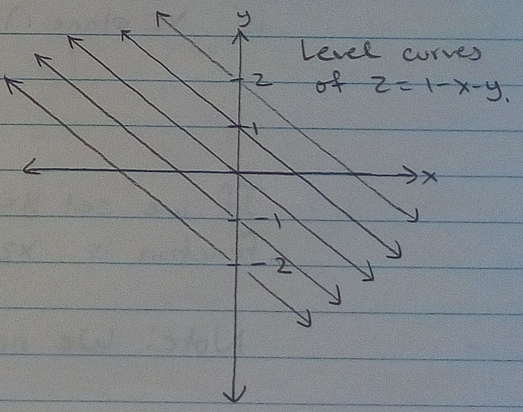
E.g. 7 Draw the level curves of the function $z = f(x,y)$ where $f(x,y) = 1-x-y$.

Solution:

Let $k = 1-x-y$, $k \in \mathbb{R}$

Choose various values for k and solve for x and y.

$k=0 \to y = -x+1$
$k=1 \to y = -x$
$k=2 \to y = -x-1$
$k=-1 \to y = -x+2$
$k=-2 \to y = -x+3$



- E.g. 8 Draw the level curves of the function $z = f(x,y)$, where $f(x,y) = \sqrt{9-x^2-y^2}$.

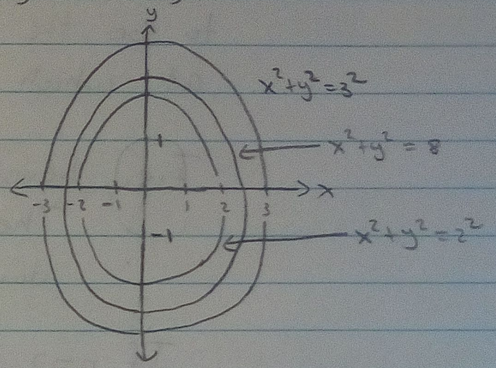
Solution: Let $k = \sqrt{9-x^2-y^2}$, $0 \le k \le 3$, $k \in R$

$k=0 \rightarrow x^2+y^2=3^2$

$k=1 \rightarrow x^2+y^2=8$

$k=2 \rightarrow x^2+y^2=5$

$k=3 \rightarrow x^2+y^2=0$



5. Method of Sections:

- By a section of the graph, we mean the intersection of the graph and a vertical plane.

- E.g. 9 If we have the function $z = x^2+y^2$, to find its sections, we have to set $x=0$ and $y=0$ (separately).

Solution:

If we set $x=0$, a section of the function is YZ plane $\cap$ $\underbrace{\{(x,y,z) \mid x=0,\ z=y^2\}}_{\text{This is the graph of } z=x^2+y^2 \text{, but with } x=0}$.

If we set $y=0$, another section of the function is XZ plane $\cap$ $\{(x,y,z) \mid y=0,\ z=x^2\}$.

Note: We never set z to 0.

- E.g. 10 The graph of the function $z = x^2 - y^2$ is a hyperbola. Draw its level curves and write its sections.

Solution:

Let $k = x^2 - y^2$, $k \in \mathbb{R}$.

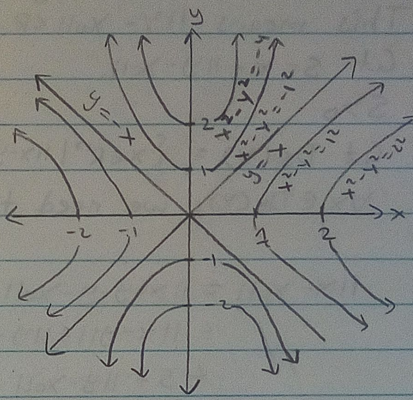
$k = -2 \rightarrow x^2 - y^2 = -2$

$k = -1 \rightarrow x^2 - y^2 = -1$

$k = 0 \rightarrow x^2 = y^2$

$k = 1 \rightarrow x^2 - y^2 = 1$

$k = 2 \rightarrow x^2 - y^2 = 2$



If we set $x=0$, we get: YZ plane $\cap \{(x,y,z) \mid x=0, z=-y^2\}$

If we set $y=0$, we get: XZ plane $\cap \{(x,y,z) \mid y=0, z=x^2\}$

6. Open Sets:

- Def: An open disk / open ball of radius r and center x_0 is the set of all points x s.t. $\|x_0 - x\| < r$. This is denoted as $D_r(x_0)$.

- Def: Let $U \subset \mathbb{R}^n$. U is an open set if for every point x_0 in U, there exists some $r > 0$ s.t. $D_r(x_0)$ is contained in U.

In $\mathbb{R}^1$, an open set is an open interval.
In $\mathbb{R}^2$, an open set is an open disk.
In $\mathbb{R}^3$, an open set is an open ball.

- Theorem: $D_r(x_0)$ is an open set.

Proof:

Let y be an arbitrary point in $D_r(x_0)$.

This means $\|y - x_0\| < R$.

Let $s = r - \|y - x_0\|$.

$s > 0$

Let $D_s(y) = \{x \in \mathbb{R}^n \mid \|x - y\| < s\}$

$\forall x \in D_s(y)$, we need to show that $\|x - x_0\| < r$.

$$\begin{aligned} \|x - x_0\| &= \|x + y - y - x_0\| \\ &\leq \|x - y\| + \|y - x_0\| \text{ (By Triangle Inequality)} \\ &< s + \|y - x_0\| \\ &< r \text{, as wanted} \end{aligned}$$

$\therefore D_r(x_0)$ is an open set.

Note: Even though the professor discussed limits this week, I will put it on my notes for next week because she said that she will go over limits in more detail in week 4 and I don't want to split the topic between 2 notes. It would be more convienent if it is together.